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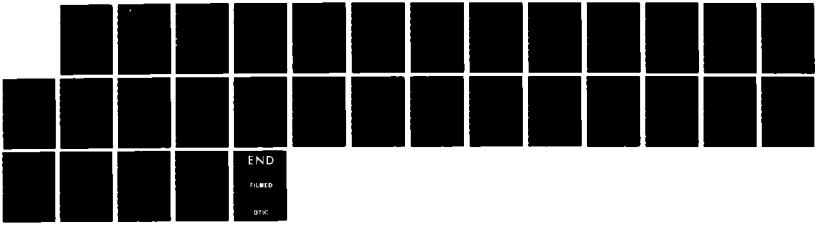

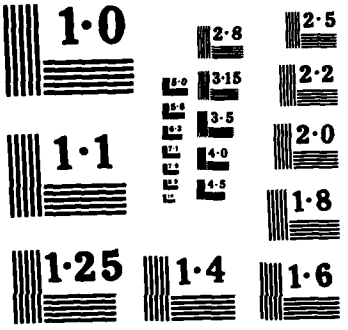

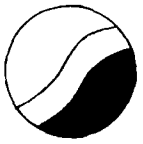

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Laboratory for Numerical Analysis

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BASIC PRINCIPLES OF FEEDBACK AND ADAPTIVE APPROACHES IN THE FINITE ELEMENT METHOD

by

I. Babuška and W. Gui

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BASIC PRINCIPLES OF FEEDBACK AND ADAPTIVE APPROACHES IN THE FINITE ELEMENT METHOD

I. Babuška*, W. Gui**

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ABSTRACT

The paper addresses the basic notions of the theory of feedback and adaptive finite element methods. An illustrative example of the finite element method for solving nonlinear differential equation of the elliptic type is presented. Numerical example of regridding process is given.

1. INTRODUCTION

The notion of feedback and adaptivity in numerical computations has attracted great attention in the recent years. All the modern ODE's solvers for initial value problems and quadratures are feedback methods. The feedback methods are increasingly in the focus of interest in the area of numerical solution of PDE and computational mechanics. Many reports, preprints and papers dealing with the adaptivity appeared recently. It seems to be that successful computation of today's large complex problems cannot be made without some kind of feedback or adaptive approach.

In general we call a numerical process a feedback process when it uses the data in a sequential way so that the flow of computation is determined by some feedback from the previous information obtained during the computation (see eg. [18] [19] [20]). However, such a feedback procedure is advantageous only if it has some useful properties, for example, if it produces the required information by less work and higher reliability than the nonadaptive process. In order to study the adaptive procedures, more precise definition of the used notions has to be introduced, see eg. [1] [3] [16].

It is worthwhile to distinguish between the feedback and adaptive procedures. A feedback procedure (method) is said to be adaptive if it is optimal with respect to certain performance measure; hence the adaptivity is a relative notion.

The analysis of a method, adaptive or nonadaptive, needs an exact description of the class of problems which the method is designed for. The comparison of a feedback and nonfeedback approach is also relative to the set of the problems under consideration. It has been shown that if the class of the problems is big enough, then in the worst case for the class (or even in the average case) the non-feedback methods are as good as the adaptive

methods. See e.g. [18] [20]. In this case one can conclude that the non-feedback method will be computationally more efficient than the feedback one (especially for parallel computation). On the other hand, as experience shows, for the majority of practically important classes of problems adaptive methods perform much better, they are more effective and reliable than the nonadaptive ones.

2. FEEDBACK AND ADAPTIVITY

We discuss now the notion of feedback and adaptivity in numerical computation in more detail. These concepts were discussed in connection with the finite element method in [1] [9]. For a more general concept see [16].

We will illustrate the introduced notions on simple example of the two point boundary value problem:

$$-(au')' + bu = f \tag{2.1a}$$

$$u(0) = u(1) = 0 \tag{2.1b}$$

where the aim is to find the approximate solution with a sufficient accuracy measured in the energy norm.

We will illustrate the introduced notions in the connection with the h-version of the finite element method (with elements degree $p = 1$).

In general, let P be the class of problems to be solved. (In our case this class is characterized by class of functions (a, b, f).)

Denote by S the set of numerical solutions. (The set of all piecewise linear functions of H_0^1.)

An algorithm is a mapping which maps the set of problems P into the set S. More precisely, an algorithm relates to an information operator N which may be referred to a set of subroutines such that the information needed by

the algorithm may be computed for the given problem P (e.g. the evaluation of a, b, f). It should be noticed that the information is, in general, only partial. (Values of a, b, f can be computed at certain points and with certain precision only.)

There are two kinds of information operator (see [18] [20]); the first is the non-feedback one (in [18] it is called non-adaptive information), which uses no information of computed results of the solution (e.g. when using a uniform mesh). The other is the feedback information operator which uses sequentially the information obtained earlier (e.g. the solution on the previously used meshes). However, no matter what kind of information is used, an algorithm produces a sequence of states (the sequence of meshes) and a sequence of corresponding solutions (the finite element solutions). The pairs of the states and solutions compose the trajectory of the algorithm. Often we also say that the sequence of the states is the trajectory. The map mapping the previous states into a new state is called a transition operator construction of the new mesh from the old ones). In the non-feedback process the transition operator of course makes no use of the previous states. The algorithm is completely described by the structure of the transition operator.

In summary a numerical procedure consists of the following components:

1) A set of problems P ((1) with the class of admissible a, b, f).

2) The set S of the numerical solutions (piecewise linear functions of H_0^1).

3) The set of states X (the set of all permissible meshes Δ).

4) A set of information I (the pointwise values of a, b, f) and an information operator N related to the algorithm $N: N \times P \times X \times S \to I$. (The subroutines which obtain the information of the problem $p \in P$ at n-th stage, $n \in \mathbf{N}$, using the meshes $\Delta_i \in X$ and FE solution $u_i \in S$, $i \leq n-1$).

5) Transition operator T, $T: N \times I \to X$. (Using information I the transition operator creates a new mesh.)

6) Solution operator I, $R: X \times I \to S$. (Obtaining the FE solution for the new mesh using information of the problem.)

7) A set of trajectories T which consists of all sequences

$$\omega = ((\Delta_0,u_0), (\Delta_1,u_1), \ldots).$$

An algorithm maps any element $P \in \mathcal{P}$ into a trajectory $\omega \in T$. We mention that we did not include (for simplicity) the stopping criterion in the process.

A feedback algorithm is useful only if it performs better than any non-feedback one for the set $\mathcal{P}$. In [1] and [14] the notion of a performance measure was introduced. A performance measure μ is a map from the set of trajectories into $\mathbb{R}^1$.

A feedback algorithm is said to be adaptive (relatively to $\mathcal{P}$, S, X, μ) if it is optimal for the performance measure μ. (In general such optimal algorithm does not have to exist for some particular $(\mathcal{P}, S, X, \mu)$.

Several concrete performance measures were introduced in [1] for the finite element method. The convergence measure μ was defined as binary measure $\mu: I \to \{0,1\}$. If the solution converges then $\mu(\omega) = 1$, otherwise $\mu(\omega) = 0$. A convergence rate measure was defined as follows: For any problem $P \in \mathcal{P}$ let $\Phi(n,P)$ be the minimal error obtainable by the mesh Δ with $m + 1$ nodal points, i.e., $\dim(S) = m$ where S is the finite element space for given mesh. The particular feedback method is optimal $\mu(\omega) = 1$ if

$$\frac{\varepsilon(\Delta_j,P)}{\Phi(m(\Delta_j),P)} < K(P), \qquad j = 0,1,2,\ldots \tag{2.2}$$

i.e., when the error ε of the finite element solution on the sequence of meshes Δ_j constructed by the algorithm is of the same magnitude as the theoretically smallest possible error Φ for all $P \in \mathcal{P}$. In [9] we described some concrete algorithms which are adaptive with respect to the convergence measure (defined above) for the set $\mathcal{P}_1$ consisting of all problems with $0 < a_0 \leq a(x) \leq a_1 < \infty$, $0 \leq b(x) \leq b_1 < \infty$ and f such that the exact solution belongs to H_0^1.

These algorithms are adaptive with respect to the convergence rate only if the set $\mathcal{P}_1$ is restricted to a smaller set $\mathcal{P}_2$. This set $\mathcal{P}_2$ is nevertheless broad enough to include vast majority of problems with practical significance. In [9] we have shown an example of $P \in \mathcal{P}_1$, $P \notin \mathcal{P}_2$ such that the algorithm produces meshes with (2.2) unbounded. Algorithm of a similar type as discussed in [9] was implemented by the code FEARS [15] and PLTMG [10] in the two dimensional setting.

In [14] we theoretically analyzed the feedback and adaptivity of some algorithm based on the h-p version. We have proven the adaptivity with respect to the convergence measure for a large class $\mathcal{P}$. The convergence rate measure was defined slightly different than before. We considered the set $\mathcal{P}_3$ of problems with solutions which are analytic except a finite number of points x_0 where the solution is of the type $|x - x_0|^\alpha$, $\alpha > .5$.

We have shown that for any $P \in \mathcal{P}_3$ and any mesh Δ and p-distribution

$$\Phi(n,P) \geq \text{Const}(P) \frac{1}{n^{\alpha/2}} r_0^{\sqrt{(\alpha-.5)n}} \tag{2.3}$$

where

$$r_0 = (\sqrt{2} - 1)^2$$

and n is the number of degrees of freedom (i.e., dimension of the used finite element space).

The algorithm which we studied and implemented in [14] leads to an error

2.4) $$\varepsilon = \text{const}(P)(\Phi(n,P))^{\gamma}$$

with γ independent of P and α. Combining (2.3) and (2.4) we immediately see that our algorithm is adaptive with respect to P_3 and obviously defined the convergence rate measure. We can also readily see that any non-feedback algorithm cannot give an exponential rate of convergence for an arbitrary problem in this set.

Various feedback algorithms appeared in the literature and are available as codes. Nevertheless there is no available analysis of the adaptive features in the sense explained above although numerical experimentation with these codes can give very good insight and lead to the proper conjectures.

3. A PARTICULAR MODEL PROBLEM

Let us be interested in the nonlinear problem

(3.1a) $$\frac{d}{dx}\, a(u',u,x) + b(u,x) = f(x,\lambda) \qquad x \in I = (0,1)$$

(3.1b) $$u(0) = u(1) = 0$$

with the aim to find the solution $u(x,\lambda)$ for $x \in I$ in the given accuracy τ in the H^1 norm for all $0 \leq \lambda \leq \lambda_0$. We will describe briefly the main parts of a feedback procedure for solving (3.1a) (3.1b). Although (3.1) is one dimensional problem, the main ideas which will be explained here are not one dimensional and they are being implemented in a two dimensional setting. The solution is computed by the continuation method using (for a fixed mesh) adaptive procedures as PITCON (see eg. [17]). The problem (3.1) has some common features with the methods for solving parabolic equations. The continuation parameter g plays analogous role as the time t in the parabolic

oblem and will be called "time" for convenience. Of course there are also sential differences here too. For more about adaptive procedures for para- lic equations we refer e.g. to [11].

In the following we shall discuss a feedback mesh regridding procedure. is clear that while the "time" parameter λ advances a fixed mesh in neral cannot work well and the mesh is necessary to be regridded. However, o frequent change of mesh also increases the cost of computation. Therefore shall study the regridding strategy (including the criterion of mesh ality) as well as the adaptivity in this procedure.

THE MESH DENSITY AND OPTIMAL MESH

We now discuss on the optimal mesh for the linearized problems. We sume that the h-version FEM is used for solving the problem, and the poly- mial degree of the elements is assumed to be p. For one dimensional case is problem was considered in [7], [13], by using the graded meshes.

A graded mesh Δ is obtained by a given grading function $g(x)$ and tensity $m(\Delta)$ (i.e., number of elements in Δ). For example, on [0,1], e nodal points of a mesh Δ are

$$x_i = g(\frac{i}{m}), \qquad i = 0,1,2,\ldots,m \tag{.1}$$

ere g is a strictly increasing continuous function with $g(0) = 0$, $g(1) =$. The optimal grading function is directly related to some derivatives of e function $u(x)$ (see e.g. [7] [8]). We have shown in [13] (see also [8] r example), that if $u(x) = x^{\alpha}$, $\alpha > .5$, the function

$$g(x) = x^{\beta}$$

$$\beta = \frac{p + .5}{\alpha - .5}$$

he optimal grading function. It is easy to show that in this case

$$\frac{d}{dx} g^{-1}(x) = C|u^{(p+1)}(x)|^{\frac{2}{2p+1}} \quad)$$

e $g^{-1}(x)$ is the inverse function to $g(x)$. The similar expression holds eneral under proper assumptions (see [7] [8]).

This approach is obviously only one dimensional. Let us modify it so it can be easily transfered into more dimensions. For this reason we l introduce the notion of the mesh density function (briefly the density).

Let $J \subset I$ be any interval and $\{\Delta_k\}$ be a sequence of meshes. Let $\bar{J})$ be the number of elements of the mesh Δ_k contained in $\bar{J}$ and ume that

$$\bar{\delta}(J) = \lim_{k\to\infty} \frac{m(\Delta_k,\bar{J})}{m(\Delta_k)} \quad 3)$$

well defined.

Function $\bar{\delta}(J)$ can be extended to a σ-additive set function and we will ume that it is differentiable. Denote by $\delta(x)$ its derivative. This ns that

$$\delta(x) = \lim_{\substack{x\in J \\ |J|\to 0}} \frac{\bar{\delta}(J)}{|J|} \quad 4)$$

sts for any $x \in I$. We will call the function $\delta(x)$ the mesh density ction for the sequence of meshes Δ_k. If the grading function exists, then is easy to see that

$$\delta(x) = \frac{d}{dx} g^{-1}(x) \quad 5)$$

hence

$$\delta(x) = C|u^{(p+1)}(x)|^{\frac{2}{2p+1}} \quad 6)$$

TABLE 1.

	λ	m	ε	χ	ψ	mesh	action	$\rho_2(k)$
1	3.0	64	0.04003	1.3599	1.1176	not O.K.	regridded	
1	3.0	48	0.04458	1.0002	1.0591	O.K.		3.7382
2	3.5	48	0.04888	1.0293	1.0114	O.K		2.4349
3	4.0	48	0.05638	1.1069	0.9417	not O.K	regridded	
3	4.0	55	0.04086	1.0058	1.1063	O.K.		3.1482
4	4.5	55	0.04222	1.0120	1.0882	O.K.		2.6894
5	5.0	55	0.04682	1.0746	1.0335	O.K.		2.4239
6	5.5	55	0.05469	1.1803	0.9562	not O.K.	regridded	
6	5.5	58	0.03977	1.0180	1.1213	O.K.		2.8321
7	6.0	58	0.03821	1.0023	1.1440	O.K.		2.6480
8	6.5	58	0.03955	1.0386	1.1244	O.K.		2.5254
9	7.0	58	0.04330	1.1188	1.0746	not O.K.	intact	2.4417
10	7.5	58	0.04900	1.2320	1.0101	not O.K.	intact	2.3844
11	8.0	58	0.05638	1.3689	0.9417	not O.K.	regridded	
11	8.0	57	0.03963	1.0673	1.1232	O.K.		2.6380
12	8.5	57	0.03452	1.0259	1.2035	not O.K.	intact	2.5877
13	9.0	57	0.03099	1.0052	1.2703	not O.K.	regridded	
13	9.0	44	0.04460	1.0076	1.0589	O.K.		2.6540
14	9.5	44	0.04247	1.0003	1.0850	O.K.		2.5827
15	10.0	44	0.04083	1.0089	1.1066	O.K.		2.5254
16	11.0	44	0.03800	1.0571	1.0147	O.K.		2.4974
			...	...	...	...		

It is possible that the new mesh could be unsuccessful, for example, it ld cause divergence in the iteration of finding the solution, or it could e a solution with low accuracy. In these cases one has to adjust the h. We will not go into detail of the adjusting procedure.

According to our assumption, when considering a process which constructs imal mesh at each step, the ratio $\rho_2(k) = 1 + \theta = 3$. Our feedback

particular let

1.2) $$D_0(x,\lambda) = \frac{10}{1 + 10[x - \sin \lambda/10]^2}$$

i select the following parameters steering the described feedback process:

error tolerance	τ	= .05
maximal mesh quality indicators	ψ_{max}	= 1.2
	χ_{max}	= 1.1
the cost control parameters	χ_1	= 2
	χ_2	= 1.2
the cost exponent	μ	= 2
the cost regridding factor	θ	= 2
the maximal number of predicted steps	k_{max}	= 10

ıe mesh is constructed in a binery tree structure to simulate data structure ıed in the two dimensional setting. This means that any node of the mesh can : obtained by a successive bisection of elements. The base mesh which is :xed in the regridding process has level 3 and hence it consits of 8 ıterval of the size 1/8.

The results are shown in Table 1. We see that the regridding is trigered ' various reasons

a) The error is not acceptable ($\psi < 1$): see mesh No. 3, 6, 11.

b) The mesh shape is bad ($\chi > \chi_{max}$): see mesh No. 1.

c) The mesh intensity is too high ($\psi > \psi_{max}$): see mesh No. 13.

Sometimes the mesh quality indicators exceed the given criteria but the ıst analysis shows that the mesh is better being kept intact, see mesh No. 9,), 12.

ntensity of the optimal mesh at λ (to achieve the given accuracy τ). et λ_j, $j = 1,2,\ldots$ be the continuation steps produced by PITCON, for xample. The ideal total cost (in k steps) will be

$$C^i_{total}(k) = \sum_{j=0}^{k} C(m_{0,\tau}(\lambda_j)). \tag{10.4}$$

We will consider the ratio

$$\rho_2(k) = \frac{C^a_{total}(k)}{C^i_{total}(k)}, \tag{10.5}$$

ınd the measure $\mu_2 = 1$ if $\rho_2(k) < \gamma$ (for some given $\lambda > 1$) for all $: > k_0$, $\mu_2 = 0$ otherwise.

We can also consider the measure with respect to the time inverval $[\Lambda_1,\Lambda_2]$ in an analogue way.

The important question is whether $\rho_2(k) < 1 + \theta$, i.e., the cost of our feedback is less than the cost when the mesh is changed to be optimal at each step.

The adaptivity of the proposed feedback can be analyzed under various stringent assumptions. Nevertheless we will not go into this analysis. Instead, we show a numerical example computed by a code we wrote.

11. A NUMERICAL EXAMPLE

As we have seen, the entire process is characterized by the (unnormalized) density function $D_0(x,\lambda)$. We can simulate the process of regridding assuming that the degree of element p and $D_0(x,\lambda)$ is given. Then for $p = 2$ the error indicator $\eta(I)$ is given by

$$\eta^2(I) = I^4 \int_I [D_0(x,)]^5 dx. \tag{11.1}$$

To define properly the measure μ_1, we have to define exactly the cost. There are various possibilities. We will adopt the following definitions:

Let λ_j, $j = 1,2,\ldots$ be the continuation steps, Δ_j be the meshes at λ_j. These meshes are not necessarily different, we assume that the regriddings only occur at λ_{j_ν}, $\nu = 1,2,\ldots$. Then we define the total cost in k continuation steps to be

$$C_{total}(k) = \sum_{j=1}^{k} (1 + \alpha_j \theta) C(m_j) \tag{10.1}$$

where $\alpha_j = 1$ if $j = j_\nu$ (i.e., regridding occurs), $\alpha_j = 0$ otherwise, and $m_j = m(\Delta_j)$ is the intensity of mesh Δ_j.

We can also consider the cost in certain time interval $[\Lambda_1, \Lambda_2]$:

$$C_{total}(\Lambda_1, \Lambda_2) = \sum_{\Lambda_1 \le \lambda_j \le \Lambda_2} (1 + \alpha_j \theta) C(m_j). \tag{10.2}$$

The cost of the feedback and non-feedback methods will be denoted by superscripts a and n respectively, then we define

$$\rho_1(k) = \frac{C^a_{total}(k)}{C^n_{total}(k)} \tag{10.3}$$

and $\mu_1 = 1$ if $\rho_1(k) \le \gamma$ (for some given $0 < \gamma < 1$) for all $k \ge k_0$, $\mu_1 = 0$ otherwise.

An analogue of (10.3) may also be made for the total cost defined by (10.2).

Another measure μ_2 can compare the cost of the introduced feedback procedure with the ideal cost of the method when optimal meshes are used at every step (neglecting the cost of regridding). Let $m_{0,\tau}(\lambda)$ be the

10. THE ADAPTIVITY OF THE REGRIDDING PROCESS

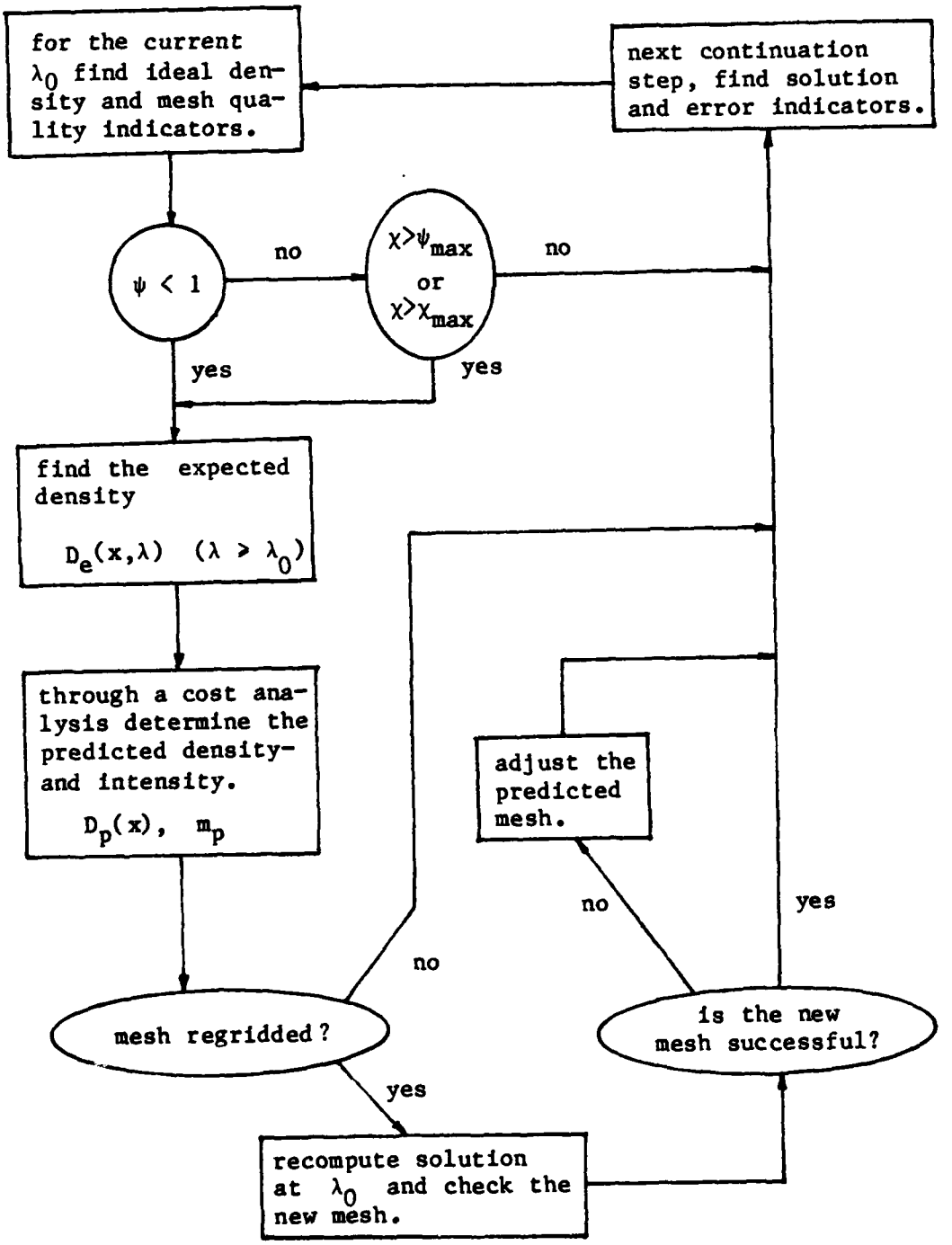
It is clear that the process described in Section 9 is a feedback process. We will first include it into the frame discussed in Section 2. The set of problems is defined by the set of functions a, b, f in (3.1a). The solution set is the finite element space of elements of degree p on the meshes Δ. The states are the pairs (λ,Δ) where Δ is the mesh at time λ. The information operator consists of a set of subroutines which evaluate the involved functions to obtain necessary information described in the last section. The transition operator makes decision of regridding and construct new meshes when necessary. The solution operator then computes the finite element solution on the given mesh λ.

In solving the nonlinear problem by continuation (for example, by PITCON), another feedback process is involved. We shall not address here the question of adaptivity of this continuation process.

The regridding process itself is very complex. For the adaptivity analysis various performance measures can be considered. Numerical experiments can be useful for formulation some conjectures for theoretical investigations.

Like the convergence measure discusssed earlier, we introduce here a natural performance measure μ_0 such that if all solutions in the trajectory satisfy given tolerance τ (which may be restricted to some range, say 1%-10%), then $\mu_0 = 1$, otherwise $\mu_0 = 0$. The algorithm is said to be adaptive with respect to μ_0 if $\mu_0 = 1$ for all problems $P \in \mathcal{P}_0$. It is clear that our algorithm is adaptive for a large class of problems $\mathcal{P}_0$.

Another natural performance measure compares the cost of our feedback procedure with the cost of the computation when fixed mesh is used. We define the measure μ_1 so that $\mu_1 = 1$ if the cost of the feedback is not larger than the cost of the fixed mesh approach. Otherwise $\mu_1 = 0$.


for the current λ_0 find ideal density and mesh quality indicators.
next continuation step, find solution and error indicators.
$\psi < 1$
no
$\chi > \psi_{max}$ or $\chi > \chi_{max}$
no
yes
yes
find the expected density $D_e(x,\lambda)$ $(\lambda > \lambda_0)$
through a cost analysis determine the predicted density and intensity. $D_p(x)$, m_p
adjust the predicted mesh.
no
yes
no
mesh regridded?
is the new mesh successful?
yes
recompute solution at λ_0 and check the new mesh.

(8.4) $$C_i = C(m_{0,\tau})$$

where $m_{0,\tau}$ is the intensity of optimal mesh at λ_0.

Assuming that the average step $\Delta\lambda$ for λ (which is selected in a feedback fashion) will stay the same as in the past, we will compute a density $D_p^{(\lambda_0,\lambda_1)}(x)$ and the corresponding intensity $m^{(\lambda_0,\lambda)}$ as in the last section for each $\lambda = \lambda_0 + k\Delta\lambda$, $k = 0,1,2,\ldots$. The predicted cost for this mesh is then

$$C_p^k = C(m^{(\lambda_0,\lambda)}) \qquad (\lambda = \lambda_0 + k\Delta\lambda).$$

This procedure is proceeded until $k = k'$:

$$C_p^{k'} > \min\{C_c/\kappa,\ \kappa_2 C_i\},$$

and

$$\kappa = \kappa_1 \frac{C_i^{old}}{C_i},$$

and κ_1, $\kappa_2 > 1$ are given a-priori, C_i^{old} is the ideal cost when last mesh regridding occurred. The predicted mesh will be constructed from the density $D_p^{(\lambda_0,\lambda)}(x)$ and $m^{(\lambda_0,\lambda)}$ with $\lambda = \lambda_0 + (k' - 1)\Delta\lambda$,

Roughly speaking, the meaning of the above selection is that the cost of a new mesh should not exceed too much the cost of the ideal mesh and should take into account the change of the optimal mesh.

9. THE MAIN FLOW OF REGRIDDING PROCESS

Here we briefly show the main flow of the feedback algorithm for the design of the mesh.

6) Make a cost analysis for determining λ_1. (See Section 8.)

7) Compute the uncertainty function $\varphi(x,\lambda)$ of the prediction by comparing the expected density with the ideal one (after being computed).

8) Design the new mesh if the current mesh has to be changed.

8. THE COST ANALYSIS

The cost of the adaptive mesh consists of two parts:

1) The cost of computation by the continuation method (for example by PITCON) per step:

$$C = C(m) = m^{\mu} . \tag{8.1}$$

where $C(m)$ is the cost function which is supposed to be only depending on the intensity.

2) The cost of regridding the mesh which will be assumed as a multiple of the cost of the solution:

$$C_0 = \Theta C(m). \tag{8.2}$$

A mesh is considered to be changed if either it cannot give the required accuracy or it costs too much (although the error is admissible). The first case is characterized by $\psi < 1$, and the latter is determined by the criteria $\chi > \chi_{max}$ or $\chi > \psi_{max}$ when χ_{max} and ψ_{max} are given a-priori.

We define the cost for the current mesh (at λ_0)

$$C_c = \begin{cases} \infty & \text{if } \chi < 1, \\ C(m_c) & \text{otherwise,} \end{cases} \tag{8.3}$$

where m_c is the intensity of the current mesh.

Furthermore, the ideal cost C_i (at λ_0) is given by

turbance of an optimal mesh causes changes of accuracy of second degree (see [7]), we do not need to know the optimal density too accurately. It is computationally advantageous to work only with meshes which have density given on the base mesh. For example, if the elements of degree $p = 2$ are used for the computation of the solution, we will assume that the density is given by the same 2 degree element function on the base mesh and the density is given by its values in the nodal points of the base mesh.

The prediction of the density goes as follows:

1) Define piecewise constant function $\tilde{D}_0(x,\lambda)$, $\lambda < \lambda_0$ on the mesh Δ so that

$$\tilde{D}_0(x,\lambda) = \frac{\eta(J)^{\frac{1}{2p+1}}}{|J|}, \qquad x \in J$$

2) By a fitting procedure construct the (unnormalized) density function $D_0(x,\lambda)$, called ideal density, on the base mesh.

3) Given function $D_0(x,\lambda)$ for $\lambda < \lambda_0$ (which is given by its values in the nodal points of the base mesh), extrapolate it for $\lambda > \lambda_0$ (by an extrapolation technique which preserves, for example, its positivity), and denote it by $D_e(x,\lambda)$ (called the expected density).

4) A new mesh, called predicted mesh, is designed for $\lambda_0 < \lambda < \lambda_1$, by (6.2)

$$D_p^{(\lambda_0,\lambda_1)}(x) = \max_{\lambda_0<\lambda<\lambda_1} D_e(x,\lambda).$$

5) Compute the intensity $m^{(\lambda_0,\lambda_1)}$ leading to the prescribed accuracy using mesh $D_p^{(\lambda_0,\lambda_1)}(x)$

$$m^{(\lambda_0,\lambda_1)} = \max_{\lambda_0<\lambda<\lambda_1} \tau^{-\frac{1}{p}} \left(\int D_p^{(\lambda_0,\lambda_1)}(x)dx\right) \left[\int \frac{(D_e(x,\lambda))^{2p+1}}{(D_p^{(\lambda_0,\lambda_1)}(x))^{2p}} dx\right]^{1/2p}$$

and

$$\delta_0^{(\lambda_0,\lambda_1)}(x) = \frac{D_0^{(\lambda_0,\lambda_1)}(x)}{\omega_0^{(\lambda_0,\lambda_1)}}. \tag{6.3}$$

Furthermore, the intensity used in $[\lambda_0,\lambda_1]$ will be determined by

$$m^{(\lambda_0,\lambda_1)} = \max_{\lambda_0\le\lambda\le\lambda_1} \frac{\omega_0^{1+1/2p}}{\tau^{1/p}} \left(\int_I \frac{(\delta_0(x,\lambda))^{2p+1}}{(\delta_0^{(\lambda_0,\lambda_1)}(x))^{2p}}\,dx\right)^{1/2p}.$$

7. DENSITY PREDICTION

The problem (3.1) is solved by a continuation method with respect to the parameter λ. This means that we know the approximate solution for $\lambda \le \lambda_0$ and want to predict the behaviour of the solution for $\lambda > \lambda_0$ and design an optimal mesh for $\lambda_0 < \lambda < \lambda_1$, where λ_1 will be properly chosen. The prediction process is essentially a pattern recognition which is composed by learning, classification, prediction and correction. Some ideas of pattern recognition were used in [11] where the concepts of the shape and intensity of a mesh were introduced.

As we have seen above the shape of the mesh is determined by its density. The function $D_0(x,\lambda)$ defined by (4.10) is an optimal density up to a multiplicative factor $\frac{1}{\omega_0}$. It also contains the information about optimal intensity using (5.5) with its approximate version

$$m_{0,\tau} = \tau^{-\frac{1}{p}} \cdot \omega_0^{1+\frac{1}{2p}} = \tau^{-\frac{1}{p}} \left(\int_I (D_0(x))^{2p+1}\right)^{1+\frac{1}{2p}}.$$

We will approximate $D_0(x)$ by a (finite element) function on a relatively coarse mesh called base mesh which does not change with λ. Because a per-

Using the formulae of Section 4 we get

$$m_\tau^{2p}\tau^2 \approx \omega_0^{2p+1} \int_I \frac{(\delta_0(x))^{2p+1}}{\delta(x)^{2p}}\,dx \tag{5.3}$$

$$m_c^{2p}\varepsilon^2 \approx \omega_0^{2p+1} \int_I \frac{(\delta_0(x))^{2p+1}}{\delta(x)^{2p}}\,dx \tag{5.4}$$

$$m_{0,\tau}^{2p}\tau^2 \approx \omega_0^{2p+1}. \tag{5.5}$$

Hence we have

$$\psi \approx \left(\frac{\tau}{\varepsilon}\right)^{1/p} \approx \left[\frac{\tau}{\left(\sum_{2p+1} \eta^2(J)\right)^{1/2}}\right]^{1/p} \tag{5.6}$$

and

$$\chi \approx \left(\int_I \frac{(\varepsilon_0(x))^{2p+1}}{(\delta(x))^{2p}}\,dx\right)^{1/2p}. \tag{5.7}$$

6. OPTIMAL MESH FOR SIMULATANEOUS APPROXIMATION

In previous sections we discussed the problem of the optimal mesh for approximating $u(x,\lambda)$ for <u>one</u> fixed value of λ. We would like to get an optimal mesh for set of functions $u(x,\lambda)$, $\lambda \in (\lambda_0,\lambda_1)$. To this end we could use (4.14) and minimize

$$\max_{\lambda_0\le\lambda\le\lambda_1} \omega_0^{2p+1}(\lambda) \int_I \frac{(\delta_0(x,\lambda))^{2p+1}}{(\delta(x))^{2p}}\,dx \tag{6.1}$$

among all densities $\delta(x)$. To find such $\delta(x)$ is not easy. Hence approximately we will take

$$D_0^{(\lambda_0,\lambda_1)}(x) = \max_{\lambda_0\le\lambda\le\lambda_1} D_0(x,\lambda) \tag{6.2}$$

$$\delta_{00}(x) = \frac{\delta_0 + \varphi_0}{1 + \varphi_0}. \tag{4.15}$$

For large uncertainty $\varphi_0 \gg 1$ we get by this technique a nearly uniform mesh.

Finally let us mention that under sufficiently strict assumptions our heuristic reasoning could be given a rigorous base. In addition, we underline that although we explained the approach in one dimension, it is valid in more dimensions, too.

5. THE MESH QUALITY INDICATORS

In the feedback process we would like to use one mesh for largest possible range of λ because of the cost consideration. This leads to the need of quantitative expression of the quality of the mesh. We will characterize the quality by two indicators. The first one is related to the intensity and second to the density.

First, we let

$$\psi = \frac{m_c}{m_\tau} \tag{5.1}$$

where m_c is the intensity of the current mesh and m_τ the intensity of the mesh, having the same density, achieving (exactly) $\varepsilon(\Delta) = \tau$. For determining m_τ (4.14) is used. If $\varepsilon(\Delta) > \tau$, then the mesh is not acceptable and we have $\psi < 1$. If $\psi \gg 1$, then the used mesh is too fine and the cost can be too high.

We define the second indicator by

$$\chi = \frac{m_\tau}{m_{0,\tau}} \tag{5.2}$$

where $m_{0,\tau}$ is the intensity for the optimal mesh and accuracy τ. Clearly we have $\chi \geq 1$, and $\chi \gg 1$ indicates that the current mesh has a density which is too far to be optimal.

the elememts of the mesh Δ_k. Then

$$\delta_0(x) = \frac{\delta_0(x)}{\omega_0} \tag{4.11}$$

is the density of the optimal mesh where

$$\omega_0 = \int_I D_0(x)\,dx. \tag{4.12}$$

$D_0(x)$ will be called the unnormalized optimal density.

Let us assume that the global error $\varepsilon^2(\Delta)$ can be written in the form

$$\varepsilon^2(\Delta) = \sum_{J\in\Delta} \eta^2(J). \tag{4.13}$$

Then for the density $\delta(x)$ and the intensity $m = m(\Delta)$ we have the relation

$$\lim_{m\to\infty} m^{2p}\varepsilon^2(\Delta) = \omega_0^{2p+1} \int_I \frac{(\delta_0(x))^{2p+1}}{\delta(x)^{2p}}\,dx. \tag{4.14}$$

(4.14) expresses the error for the mesh Δ with the density $\delta(x)$ through the optimal density. Obviously the choice $\delta(x) = \delta_0(x)$ minimizes the expression on the right hand side of (4.14) and gives the minimal value ω_0^{2p+1} (because $\int \delta_0(x)dx = 1$). In practice we express $\delta_0(x)$ by the error indicators and (4.10) (4.11). Nevertheless, the error indicators (especially in more dimensions) are showing a dispersion and a smoothing technique has to be used. In addition it is useful to assume that we have in (4.14) for disposition $\delta_0(x) + \varphi(x)$ instead $\delta_0(x)$ only where $\varphi(x)$ expresses the uncertainty in the determination of $\delta_0(x)$ which can be determined by the smoothing process mentioned above; for example, we can assume that $|\varphi(x)| \leq \varphi_0$. Now the optimal mesh under this uncertainty minimizes (4.14) for the worst case of the uncertainty. This leads to the density $\delta_{00}(x)$

with the constant C such that

$$\int_I \delta(x)\,dx = 1. \tag{4.7}$$

The notion of the density function can be obviously easily extended to n-dimensions.

If the density function is given, then it is not hard to construct a mesh with this density $\delta(x)$ and given intensity $m(\Delta)$. We shall assume that all meshes we are dealing with have a density function.

Let us assume now that an error indicator $\eta(J)$ of every element $J \in \Delta$ of the mesh is computable and it is such that

$$\eta(J) = \|e\|_{E(J)}(1 + o(1)) \tag{4.9}$$

where $\|e\|_{E(J)}$ is the (local) error (in the energy norm) of the interpolant of u on J. Then we get

$$\lim_{\substack{|J|\to 0 \\ x\in J}} \frac{\eta^2(J)}{|J|^{2p+1}} = C|u^{(p+1)}(x)| \tag{4.9}$$

with a-priori known constant C. See [8] for more details. Similar situation occurs in more dimensions. For the error indicators in two dimension we refer to [2].

The error indicator gives through (4.9) the information about the derivative of the approximated function u and hence also the density for the optimal mesh. To this end we define

$$D_0(x) = \lim_{\substack{|J|\to 0 \\ x\in J}} \frac{\eta(J)^{\frac{1}{2p+1}}}{|J|} \tag{4.10}$$

and assume that $D_0(x)$ can be well approximated by the error indicators of

regridding process obtains the ratio $\rho_2(k) \approx 2.5$ (see Table 1). Thus it is more efficient that the previous one.

The process we have shown is of course not the only possible approach and it needs detailed theoretical and numerical studies in the spirit we explained above. We have shown it as an example of application of principles of studying the adaptivity.

12. CONCLUSIONS

We have shown the main questions associated to the design and analysis of the feedback procedures. The theoretical analysis is often not easy. Nevertheless, every feedback algorithm could be conjectured to be adaptive with respect to some measures and class of problems and studied theoretically and experimentally in the direction we explores.

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